A type system for monotonicity

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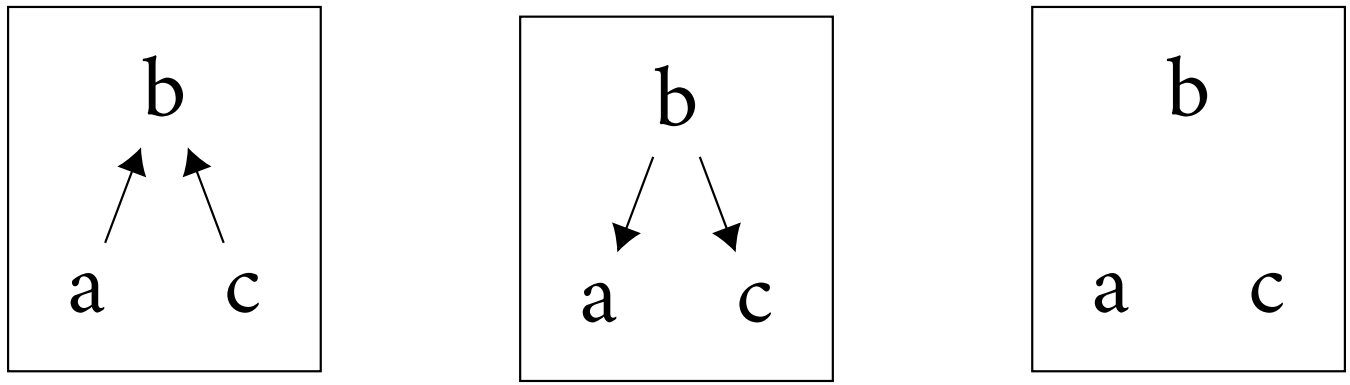
It's just the simply-typed λ-calculus!

$$\begin{aligned} [\![A]\!] &\in \textbf{Poset} \\ [\![A \to B]\!] &= \textit{monotone} \text{ maps } [\![A]\!] \to [\![B]\!], \\ &\quad \text{ordered pointwise} \end{aligned}$$

A $\quad$ $\mathrm{op}\, A$ $\quad$ $\Box A$



$f : \Box A \to B$ is monotone iff

$$x = y \implies f(x) \leqslant f(y)$$

i.e. **always**!

$$
\begin{aligned}
&\mathrm{setMap} \;:\; \Box(\Box A \to B) \to \mathrm{Set}\; A \to \mathrm{Set}\; B \\
&\mathrm{setMap}\; f\; xs = \\
&\quad \textbf{let box}\; g = f\; \textbf{in} \\
&\quad \textbf{do}\; x \leftarrow xs \\
&\qquad \textbf{let box}\; y = x \\
&\qquad \mathrm{return}\; (\textbf{box}\; (g\; (\textbf{box}\; y)))
\end{aligned}
$$

$$\text{setMap} \;:\; \Box(\Box A \to B) \to \text{Set}\; A \to \text{Set}\; B$$

$$\text{setMap}\; f\; xs =$$

$$\textbf{do}\; x \leftarrow xs$$

$$\text{return} \qquad (f \qquad x)$$

$$A <: B$$

$$\mathrm{id} : \quad A \to B$$

$$[\mathsf{T}]A <: B$$

$$\mathrm{id} : \mathsf{T}A \to B$$

$$\mathsf{T} \in \{\mathrm{id}, \mathrm{op}, \Box, \ldots\}$$

$$f : TA \to B \qquad g : UB \to C$$

$$g \circ f : (UT)A \to C$$

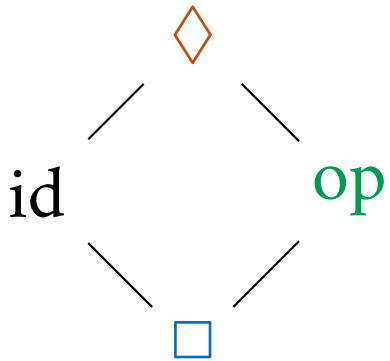
Monotonicity tames dragons!

1. Eventual consistency in distributed systems
   `http://bloom-lang.net/calm/`
2. Determinism in parallel programs
   LVars: Lattice-based Data Structures for Deterministic Parallelism, Lindsey Kuper & Ryan Newton
3. Recursive queries in Datalog & Datafun
   `http://www.rntz.net/datafun`
4. Paradoxes of self-reference

FIN

$$\text{subtractEach} \;:\; \text{List}\,(\mathbb{Z} \times \text{op}\,\mathbb{Z}) \to \text{List}\,\mathbb{Z}$$
$$\text{subtractEach}\,xs = [x - y \mid (x, y) \leftarrow xs]$$

$$
\begin{aligned}
a \leqslant b : \mathrm{id}\, A &\iff a \leqslant b : A \\
a \leqslant b : \mathrm{op}\, A &\iff a \geqslant b : A \\
a \leqslant b : \Box A &\iff a \leqslant b \wedge a \geqslant b : A \\
a \leqslant b : \Diamond A &\impliedby a \leqslant b \vee b \leqslant a : A
\end{aligned}
$$



$\sqcup_T$	id	op	$\Box$	$\Diamond$
id	id	op	$\Box$	$\Diamond$
op	op	id	$\Box$	$\Diamond$
$\Box$	$\Box$	$\Box$	$\Box$	$\Diamond$
$\Diamond$	$\Diamond$	$\Diamond$	$\Box$	$\Diamond$

($\sqcup$: T)

SUBSUMPTION

$$\frac{\Gamma \vdash M : A \qquad [T]A <: B}{T\Gamma \vdash M : C}$$